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PERFECT MAPPINGS AND CERTAIN INTERIOR IMAGES OF M-SPACES(1)

BY

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ABSTRACT. The main theorems of this paper show that certain conditions (called λ_c, λ_b, β_c, and β_b) are invariant, in the presence of T_0-regularity, under the application of closed continuous peripherally compact mappings. Interest in these conditions lies in the fact that they may be used to characterize certain regular T_0 open continuous images of some classes of M-spaces in the sense of K. Morita, and in the fact that they are preserved by open continuous mappings with certain appropriate additional conditions. For example, the authors have shown that a regular T_0-space is an open continuous image of a paracompact Čech complete space if and only if the space satisfies condition λ_b [Pacific J. Math. 37 (1971), 265–275]. Moreover, in the same paper it is shown that if a completely regular T_0-space satisfies condition λ_b then any T_0 completely regular open continuous image of it also satisfies λ_b. These results together with the results of the present paper and certain known results lead to the following theorem: The smallest subclass of the class of regular T_0-spaces which contains all paracompact Čech complete spaces and which is closed with respect both to the application of perfect mappings and to the application of open continuous mappings preserving T_0-regularity is the subclass satisfying condition λ_b. Similar results are obtained for the regular T_0-spaces satisfying λ_c, β_b, and β_c. The other classes of M-spaces involved are the regular T_0 complete M-spaces (i.e., spaces which are quasi-perfect preimages of complete metric spaces), T_2 paracompact M-spaces, and regular T_0 M-spaces. In the last two cases besides the interiority of the mappings the notion of uniform λ-completeness, which generalizes compactness of a mapping, enters. (For details see General Topology and Appl. 1 (1971), 85–100.)

The proofs are accomplished through the use of two basic lemmas on closed continuous mappings satisfying certain additional conditions.

1. **Introduction.** The main theorems of this paper display some invariants of perfect mappings and certain other closed continuous mappings having regular T_0 domains. The invariants are given in the form of certain sequence conditions λ_c,

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λ_b, β_c, β_b. These were introduced in [15], where examples and discussion may be found.

Some reasons for interest in these conditions may be elicited from the following results: I. A regular T_0-space is an open continuous image of a paracompact Čech complete space (regular T_0 complete M-space(2)) if and only if it satisfies condition λ_b (λ_c). II. A regular T_0-space is an open continuous uniformly λ-complete(3) image of a paracompact M-space (regular T_0 M-space) if and only if it satisfies β_b (β_c). (The sufficiency part of these theorems is proved in [15]. The proof of necessity is to be submitted.)

Consider the first parts of I and II. Frolík [4] showed that perfect mappings between Tychonoff spaces preserve Čech completeness and Michael [9] showed that closed continuous mappings preserve T_2 paracompactness. Filippov [2] and Ishii [6] showed that perfect mappings preserve the property of being a T_2 paracompact M-space (equivalently, a T_2 paracompact p-space). However from [14] it follows that there exists an open continuous uniformly λ-complete mapping of a complete metric space onto a space which is neither paracompact nor an M-space. With regard to the second parts of I and II, Morita gave an example [13] of a perfect mapping of a Hausdorff locally compact M-space onto a space which is not an M-space. Contrastingly, Theorem 2.1 may be applied with the propositions of I and II to obtain the following results.

Theorem 1.1. *The smallest subclass of the class of regular T_0-spaces which contains all paracompact Čech complete spaces (respectively, all regular T_0 complete M-spaces) and which is closed with respect both to the application of perfect mappings and to the application of open continuous mappings preserving T_0-regularity is the subclass satisfying condition λ_b (respectively, condition λ_c).*

Theorem 1.2. *The smallest subclass of the class of regular T_0 spaces which contains all paracompact Hausdorff p-spaces (respectively, all regular T_0-M-spaces) and which is closed with respect to both the application of perfect mappings and to the application of uniformly λ-complete open continuous mappings preserving T_0-regularity is the subclass satisfying condition β_b (respectively, condition β_c).*

Another point of interest arises from Worrell's example [22] of a p-space which is mapped onto a space which is not a p-space by a perfect mapping. The authors have introduced and elaborated the concept of a μ-space [17] in which the finite intersection property of the p-space concept is replaced with a decreasing sequence.

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(2) A topological space is said to be a *complete M-space* if and only if it is a quasi-perfect pre-image of a complete metric space. The term *M-space* is used in the sense of Morita [12].

(3) Uniformly λ-complete mappings are defined in [15].

Below it is shown that a completely regular T_0-space is a μ-space (complete μ-space) if and only if it satisfies β_b (λ_b). Thus the result that such spaces are preserved by perfect mappings with completely regular T_0 domains shows a significant difference between the concept of a μ-space and that of a p-space.

Two final remarks of a general nature remain to be made. The general topic of this paper concerns principles of uniformization for topological spaces. Examination of the definitions below shows that the sequence conditions given are variations of a fundamental uniformization principle in the first countable case: the property of being an essentially T_1 space having a base of countable order. This is clear on recalling that a space has this property if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ of collections of open sets covering the space such that if (G_n) is a sequence such that each $G_{n+1} \subset G_n$, each $G_n \in \mathcal{G}_n$, and $x \in \bigcap \{G_k : k \in N\}$, then $\{G_k : k \in N\}$ is a base at x [19]. This helps account for a close relationship in method of the present paper to methods of previous papers treating bases of countable order and developable spaces (cf. especially [19], [20], [21]) and also for the essential unity of treatment of the proofs of of the main results. The present article shows the satisfactory behavior of the principles of uniformization λ_c, λ_b, β_c, β_b under perfect mappings. Our final remark is to call attention to the important role which compactness in the sense of Fréchet [3] plays in the concepts and in the proofs of the principal results.

Notation and terminology. The notation and terminology used here conforms closely with that of Kelley [7]. If A is a set, $\mathrm{Fr}(A)$ denotes the boundary of A and $\mathrm{Int}(A)$ denotes the interior of A. The letter N denotes the set of positive integers, and letters i, j, k, and n are used to signify members of N. The notation $(U_n)_{n \in N}$ is used for a sequence, i.e., a function U with domain N; this notation will be abbreviated to (U_n). A sequence (U_n) of sets is said to be *decreasing* if and only if, for each n, $U_{n+1} \subset U_n$. A *representative* of a sequence $(\mathcal{A}_n)$ of sets is, by definition, a sequence (B_n) such that, for each n, $B_n \in \mathcal{A}_n$. A mapping is said to be *peripherally compact* if and only if the inverse image of every point has a compact boundary. A mapping is said to be *quasi-perfect* [13] (*perfect*) if and only if it is a closed continuous mapping such that the inverse image of every point is countably compact (compact). If $\mathcal{A}$ is a collection of sets $\bigcup \mathcal{A}$ denotes the union of the collection. An expression such as "$\bigcup \mathcal{F} \cup M$" denotes the union of $\bigcup \mathcal{F}$ and M.

2. **Definitions and results.** The definitions of the basic concepts and the statements of the main theorems are given here. A discussion of the conditions of Definitions 2.2 and 2.3 together with examples is given in [15].

Definition 2.1 [18]. Let $M \subset X$. A sequence $(\mathcal{C}_n)$ of collections of subsets of X covering M is said to be a *monotonically contracting sequence of M in X* if and only if, for each $n \in N$, if $x \in A \in \mathcal{C}_n$ and $x \in M$, there exists $B \in \mathcal{C}_{n+1}$ such

that $x \in B \subset A$. If $M = X$, the sequence is called *monotonically contracting*.

Definition 2.2 [15]. A topological space X is said to satisfy condition λ_c (λ_b) if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ such that: (1) Each $\mathcal{G}_n$ is a collection of open subsets of X covering X. (2) If (G_n) is a decreasing representative of $(\mathcal{G}_n)$ and each G_n is nonempty then $B = \bigcap\{\overline{G}_n : n \in N\}$ is nonempty and countably compact (compact) and U open and $B \subset U$ implies that some $G_k \subset U$.

Definition 2.3 [15]. A topological space X is said to satisfy condition β_c (β_b) if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ such that: (1) Each $\mathcal{G}_n$ is a collection of open subsets of X covering X. (2) If (G_n) is a decreasing representative of $(\mathcal{G}_n)$ then if $B = \bigcap\{\overline{G}_n : n \in N\}$ is nonempty it is countably compact (compact) and U open and $B \subset U$ implies that some $G_k \subset U$.

Theorem 2.1. *Suppose X is a regular T_0-space and there is a peripherally compact closed continuous mapping of X onto Y. If X satisfies any of the conditions λ_c, λ_b, β_c, or β_b then Y satisfies the same condition.*

Theorem 2.2. *Suppose f is a closed continuous mapping of a regular T_0-space X onto a space Y such that if U is open in Y and $y \in U$ there is a sequence (V_n) of open sets containing y such that any representative of (V_n) has a cluster point in U.*(4) *Also suppose that for each $y \in Y$, if $\mathcal{H}$ is a collection of open subsets of X covering $\mathrm{Fr}(f^{-1}(y))$, then there exists a point-countable collection of open sets covering $\mathrm{Fr}(f^{-1}(y))$ and refining $\mathcal{H}$. Then if X satisfies any of the conditions λ_c, λ_b, β_c, or β_b then Y satisfies the same condition.*

The concepts defined just below have been introduced and elaborated in [17]. The class of μ-spaces includes the class of p-spaces of Arhangel'skiĭ [1], and the class of complete μ-spaces includes the class of Čech complete spaces. The introduction of these concepts permits a more satisfactory theory of invariance under peripherally compact closed continuous mappings and certain open continuous mappings as is shown by Theorem 2.3 and the results of [15], respectively.

Definition 2.4 [17]. A T_1-space X is called a *μ-space* if and only if there exists a sequence $(\mathcal{G}_n)$ of collections of open subsets of the Wallman compactification ωX of X such that: (1) $(\mathcal{G}_n)$ is a monotonically contracting sequence of X in ωX. (2) If (G_n) is a decreasing representative of $(\mathcal{G}_n)$, then if $\bigcap\{G_n : n \in N\}$ contains a point of X it is a subset of X.

Remark 2.1. If X is a completely regular T_0-space, then X is a μ-space if and only if there is a monotonically contracting sequence $(\mathcal{G}_n)$ of X in bX of collections of open subsets of some T_2 compactification bX of X having property (2). In this case if there is such a sequence in some T_2 compactification, there is one in every T_2 compactification.

(4) I.e., Y has a base each element of which is a q-space in the sense of [10].

Remark 2.2. Lemma 3.6 gives an intrinsic characterization of completely regular T_0 μ-spaces.

Definition 2.5 [18]. A subset M of a topological space X is said to be a *set of interior condensation* in X if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ of M in X of collections of open subsets of X covering M such that if (G_n) is a decreasing representative of $(\mathcal{G}_n)$ then $\bigcap\{G_n: n \in N\} \subset M$.

Definition 2.6. A completely regular T_0-space X is called a *complete μ-space* if and only if it is a set of interior condensation in its Stone-Čech compactification.

Remark 2.3. A complete μ-space is a set of interior condensation in any of its T_2 compactifications (indeed, in any Hausdorff space in which it can be densely embedded). This justifies the alternative terminology of *set of absolute interior condensation* and shows that this generalizes the idea of Čech completeness.

Remark 2.4. Lemma 3.6 gives an intrinsic characterization of complete μ-spaces. For another characterization and further information see [18].

Theorem 2.3. *Suppose X and Y are completely regular T_0-spaces and there is a peripherally compact closed continuous mapping of X onto Y. Then if X is a μ-space (respectively, a complete μ-space) so is Y.*

Theorem 2.4. *Suppose X and Y are completely regular T_0-spaces and f is a closed continuous mapping of X onto Y. If f and Y have the properties stated in the hypothesis of Theorem 2.2, then if X is a μ-space (respectively, a complete μ-space) so is Y.*

3. **Lemmas on closed mappings; other results.** The first two lemmas below state equivalent forms of the basic conditions in the regular T_0 case which are convenient for use in the proofs. The next three lemmas are useful in several proofs. Lemma 3.6 characterizes completely regular T_0 μ-spaces and complete μ-spaces in terms of conditions β_b and λ_b respectively. Lemma 3.7 is a fundamental lemma on closed mappings. The key idea of the proof is found in the proof of the Lemma of [21]. Lemma 3.8 is the main lemma used in proving the theorems. It is similar to other lemmas used in first countable cases in [20], [21].

Lemma 3.1 [15]. *Suppose X is a regular T_0-space. Then in X the condition $\lambda_c(\lambda_b)$ is equivalent to: There exists a sequence $(\mathcal{G}_n)$ of collections of open sets covering X such that: (1) For each $n \in N$, if $x \in G \in \mathcal{G}_n$ there exists $G' \in \mathcal{G}_{n+1}$ such that $x \in G'$ and $\overline{G}' \subset G$. (2) If (G_n) is a sequence such that $\emptyset \neq \overline{G}_{n+1} \subset G_n \in \mathcal{G}_n$ for all $n \in N$, then $B = \bigcap\{G_n: n \in N\}$ is nonempty and countably compact (compact) and every open U which includes B also includes some G_n.*

Lemma 3.2 [15]. *If X is a regular T_0-space the condition $\beta_c(\beta_b)$ is equivalent to the condition obtained from that of Lemma 3.1 by replacing (2) with:*

(2′) *If* (G_n) *is a sequence such that* $\overline{G}_{n+1} \subset G_n \in \mathcal{G}_n$ *for all* $n \in N$, *then* $B = \bigcap\{G_n : n \in N\}$ *is countably compact (compact) and if* B *is nonempty every open* U *which includes* B *also includes some* G_n.

Definition 3.1. If S is a subset of a set X a sequence $(\mathcal{H}_n)$ is said to be a *primitive sequence* [16] of S in X if and only if for each $n \in N$ these conditions are satisfied: (1) $\mathcal{H}_n$ is a well-ordered collection of subsets of X covering S. (2) If $H \in \mathcal{H}_n$ there exists $x \in H \cap S$ not in any predecessor of H. (3) If $x \in S$, $n < k$, and H and H' are the first elements of $\mathcal{H}_n$ and $\mathcal{H}_k$, respectively, that contain x, then $H \supset H'$.

Lemma 3.3 [16]. *If* S *is a subset of a set* X *and* $(\mathcal{G}_n)$ *is a monotonically contracting sequence of* S *in* X, *there exists a primitive sequence* $(\mathcal{H}_n)$ *of* S *in* X *such that each* $\mathcal{H}_n \subset \mathcal{G}_n$.

The following lemma formalizes an argument used repeatedly in a number of publications of the authors.

Lemma 3.4. *Suppose* $(\mathcal{H}_n)$ *is a primitive sequence of* S *in* X *and* (G_n) *is a decreasing sequence of sets such that for each* n *there exists* $y_n \in G_n \cap S$ *such that the first element of* $\mathcal{H}_n$ *that contains* y_n *includes* G_n. *Then there exists a decreasing representative* (H_n) *of* $(\mathcal{H}_n)$ *such that, for each* n, H_n *is the first element of* $\mathcal{H}_n$ *that includes a term of* (G_n).

Proof. For each n there exists a first $H_n \in \mathcal{H}_n$ that includes a term of (G_n). For each n there exists $j > n + 1$ such that $G_j \subset H_n \cap H_{n+1}$. Let H denote the first element of $\mathcal{H}_n$ that contains y_j. Since $(\mathcal{H}_n)$ is a primitive sequence, H includes the first element H' of $\mathcal{H}_j$ that contains y_j. Thus $H \supset H' \supset G_j$. Therefore H does not precede H_n. Since $y_j \in H_n$ it follows that $H = H_n$. Similarly H_{n+1} is the first element of $\mathcal{H}_{n+1}$ that contains y_j. Therefore $H_{n+1} \subset H_n$.

Definition 3.2 and Lemma 3.5 are introduced only to shorten certain proofs.

Definition 3.2. A sequence (V_n) of subsets of a topological space has *property* B if and only if $\bigcap\{\overline{V}_n : n \in N\}$ is nonempty and countably compact and U open and $U \supset \bigcap\{\overline{V}_n : n \in N\}$ implies that some $V_k \subset U$.

Lemma 3.5. *Suppose* X *is a* T_1*-space and* (B_n) *is a decreasing sequence of subsets of* X *with property* B. *Suppose* (A_n) *is a decreasing sequence of nonempty subsets of* X *such that for each* n *some* $A_j \subset B_n$. *Then* (A_n) *has property* B. (*A similar lemma is proved in* [15].)

Lemma 3.6. *Suppose* X *is a completely regular* T_0*-space. Then* X *is a* μ*-space (complete* μ*-space)*(5) *if and only if* X *satisfies condition* $\beta_b(\lambda_b)$.

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(5) The part of Lemma 3.6 relating to complete μ-space may be proved from Theorem 4 of [18] and Lemma 3.1.

Proof. Let βX denote the Stone-Čech compactification of X and if $A \subset X$ ($A \subset \beta X$) let $\overline{A}$ ($\mathrm{cl}(A)$) denote the closure of A in X (in βX). Suppose X satisfies λ_b or β_b. Then there exists a monotonically contracting sequence $(\mathcal{G}_n)$ satisfying the appropriate condition of Definition 2.2 or 2.3 relative to X. By Lemma 3.3, there exists a primitive sequence $(\mathcal{H}_n)$ of X in X such that each $\mathcal{H}_n \subset \mathcal{G}_n$. There exists a function D on $\mathcal{H}_1$ such that if $U \in \mathcal{H}_1$, $D(U)$ is an open subset of βX such that $D(U) \cap X = U$. Let $\mathcal{W}_1$ denote $\{D(U): U \in \mathcal{H}_1\}$. Suppose $\mathcal{W}_1, \dots, \mathcal{W}_n$ have been defined and are collections of open sets of βX covering X such that, for $k < n$, $x \in W \in \mathcal{W}_k$ and $x \in X$ implies the existence of $W' \in \mathcal{W}_{k+1}$ such that $x \in W' \subset W$. Let $\mathcal{W}_{n+1}$ denote the collection of all sets of the form $V \cap U$ such that $U \in \mathcal{W}_n$, V is open in βX, $V \cap X \in \mathcal{H}_{n+1}$, and U contains a point of $V \cap X$ not in any predecessor of $V \cap X$ in $\mathcal{H}_{n+1}$. Then $(\mathcal{W}_n)$ is a monotonically contracting sequence of collections of open subsets of βX covering X. For $\mathcal{W}_1$ covers X. Suppose $\mathcal{W}_n$ covers X and $x \in U \in \mathcal{W}_n$. There exist a first $H \in \mathcal{H}_{n+1}$ that contains x and an open V in βX such that $V \cap X = H$. Therefore $x \in U \cap V \in \mathcal{W}_{n+1}$. Suppose (W_n) is a decreasing representative of $(\mathcal{W}_n)$. For each n, there exists $x \in W_n \cap X$ such that the first element of $\mathcal{H}_n$ that contains x includes $W_n \cap X$. Therefore there exists, by Lemma 3.4, a decreasing representative (H_n) of $(\mathcal{H}_n)$ such that, for each n, H_n is the first element of $\mathcal{H}_n$ that includes a term of $(W_n \cap X)$. Suppose β_b holds and there exists $x \in \bigcap\{W_n: n \in N\} \cap X$. Then $x \in \bigcap\{H_n: n \in N\}$. Since each $H_n \in \mathcal{G}_n$, (H_n) has property B (see Definition 3.2) and $\bigcap\{\overline{H}_n: n \in N\}$ is compact. Therefore, by Lemma 3.5, $(W_n \cap X)$ has property B. Similarly if λ_b holds there is an $x \in \bigcap\{W_n: n \in N\} \cap X$ and $(W_n \cap X)$ has property B. Suppose in either case that $z \in \bigcap\{W_n: n \in N\}$ and $z \notin X$. Then $z \notin E = \bigcap\{\overline{W_n \cap X}: n \in N\}$. Since $E \subset \bigcap\{\overline{H}_n: n \in N\}$ it is compact. Hence there is a set D open in βX such that $E \subset D$ and $z \notin \mathrm{cl}(D)$. There exists k such that $W_k \cap X \subset D \cap X$. Therefore $\mathrm{cl}(W_k) = \mathrm{cl}(W_k \cap X) \subset \mathrm{cl}(D \cap X) = \mathrm{cl}(D)$. But this implies $z \in \mathrm{cl}(D)$, which involves a contradiction. Hence $\bigcap\{W_n: n \in N\} \subset X$ so that X is a μ-space or a complete μ-space accordingly as β_b or λ_b holds.

Now suppose X is nonempty and either a μ-space or a complete μ-space. Then there exists a monotonically contracting sequence $(\mathcal{G}'_n)$ of open subsets of βX covering X and satisfying the condition of Definition 2.4 or 2.5. By Lemma 3.3 there exists a primitive sequence $(\mathcal{H}_n)$ of X in βX such that each $\mathcal{H}_n$ covers X and is a subcollection of $\mathcal{G}'_n$. In [16] it is shown that under these conditions there exists a primitive sequence $(\mathcal{W}_n)$ of X in βX of collections of nonempty open subsets of βX covering X such that: (1) If $n < k$ and $x \in X$, then if W and W' are the first elements of $\mathcal{W}_n$ and $\mathcal{W}_k$, respectively, that contain x and H is the first element of $\mathcal{H}_n$ that contains x, then $\mathrm{cl}(W') \subset W \cap H$. Let $\mathcal{C}_1$ denote $\mathcal{W}_1$. If $n > 1$, let $\mathcal{C}_n$ denote the collection of all sets of the form $U \cap W$ where $W \in \mathcal{W}_n$, U is open in βX and contains a point of $W \cap X$ not in any predecessor of W in $\mathcal{W}_n$,

and, for some $W' \in \mathcal{C}_{n-1}$, $\mathrm{cl}(U) \subset W'$. For each n let $\mathcal{G}_n = \{C \cap X: C \in \mathcal{C}_n\}$. Then each $\mathcal{G}_n$ covers X. Suppose $x \in G \in \mathcal{G}_n$. There exists $C \in \mathcal{C}_n$ such that $C \cap X = G$. Let W be the first element of $\mathfrak{W}_{n+1}$ that contains x. There exists an open subset U of βX such that $x \in U$ and $\mathrm{cl}(U) \subset C$. Then $x \in U \cap W \in \mathcal{C}_{n+1}$ and $\mathrm{cl}(U \cap W) \subset C$. Hence $U \cap W \cap X \in \mathcal{G}_{n+1}$ and $\overline{U \cap W \cap X} \subset G$. Thus condition (1) of Lemmas 3.1 and 3.2 is satisfied. Suppose (G_n) is a sequence such that $\overline{G}_{n+1} \subset G_n \in \mathcal{G}_n$ for all n. Then by condition (1) on $(\mathfrak{W}_n)$ there exists a representative (W_n) of $(\mathfrak{W}_n)$ such that, for each n, W_n is the first element of $\mathfrak{W}_n$ that includes a term of (G_n). With the use of an argument similar to the proof of Lemma 3.4 it follows that $\mathrm{cl}(W_{n+1}) \subset W_n$. By Lemma 3.4 there exists a decreasing representative (H_n) of $(\mathcal{H}_n)$ such that, for each n, H_n is the first element of $\mathcal{H}_n$ that includes a term of (W_n). Suppose there exists $x \in \bigcap\{G_n: n \in N\}$. Then $\bigcap\{H_n: n \in N\} \subset X$ since X is a μ-space. Therefore $\bigcap\{W_n: n \in N\}$ is compact and has property B. Thus $\bigcap\{G_n: n \in N\} = \bigcap\{\overline{G}_n: n \in N\} = \bigcap\{\mathrm{cl}(G_n): n \in N\}$ is compact and has property B by Lemma 3.5 and X satisfies β_b by Lemma 3.2. If X is a complete μ-space then $\bigcap\{H_n: n \in N\} \subset X$ and the argument just used together with the assumption that each $G_n \neq \emptyset$ shows that $\bigcap\{G_n: n \in N\}$ is compact and has property B. Thus X satisfies λ_b by Lemma 3.1.

Lemma 3.7. *Suppose f is a closed continuous mapping of a T_1-space X onto a space Y. Suppose $y \in Y$ is such that for any open U containing y there is a sequence (V_n) of open sets containing y such that any representative of (V_n) has a cluster point in U and if $\mathcal{H}$ is a collection of open sets in X covering $B = \mathrm{Fr}(f^{-1}(y))$ there exists a point-countable collection of open sets covering B and refining $\mathcal{H}$. Then every collection of open sets covering B includes a finite subcollection $\mathcal{F}$ such that $(\bigcup \mathcal{F}) \cup f^{-1}(y)$ is open. If X is also regular, then B is compact.*

Proof. Let M denote $f^{-1}(y)$. If the conclusion is not satisfied there exists a point-countable collection $\mathcal{G}$ of open sets covering B such that $\bigcup \mathcal{H} \cup M$ is not open for any finite $\mathcal{H} \subset \mathcal{G}$. For each $x \in \bigcup \mathcal{G}$ let $\alpha(x)$ denote an infinite sequence such that G is a term of $\alpha(x)$ if and only if $x \in G \in \mathcal{G}$. Since f is closed, there exists an open set D in X such that $D = f^{-1}f(D)$ and $M \subset D \subset \bigcup \mathcal{G} \cup \mathrm{Int}\, M$. Therefore $y \in f(D)$. Let (V_n) denote a decreasing sequence as described in the hypothesis relative to y and $f(D)$ with $V_1 \subset f(D)$. There exists a sequence (x_n) such that $x_1 \in (f^{-1}(V_1) \setminus M) \cap \bigcup \mathcal{G}$ and, for each $n > 1$,

$$x_n \in [(f^{-1}(V_n) \setminus M) \cap \bigcup \mathcal{G}] \Big\backslash \bigcup_{i=1}^{n-1} f^{-1}[\{f(x_i)\}]$$

and, for each n, x_{n+1} is not in any of the first n terms of any of $\alpha(x_1), \cdots, \alpha(x_n)$. Since M is not open, there exists $x_1 \in f^{-1}(V_1) \setminus M$. Assume $x_1, \cdots, x_n$ have been defined. Let $\mathcal{F}$ denote the collection of the first n terms of each of

$\alpha(x_1), \cdots, \alpha(x_n)$. Then $\bigcup \mathcal{F} \cup M$ is not open. There exists $G \in \mathcal{G}$ such that

$$A(G) = (f^{-1}(V_{n+1}) \cap G) \Big\backslash \bigcup_{i=1}^{n} f^{-1}[\{f(x_i)\}] \not\subset \bigcup \mathcal{F} \cup M.$$

For otherwise $B \subset \bigcup\{A(G): G \in \mathcal{G}\} \subset \bigcup \mathcal{F} \cup M$ which implies that $\bigcup \mathcal{F} \cup M$ is open. Select $x_{n+1} \in A(G)\backslash(\bigcup \mathcal{F} \cup M)$. The sequence $(f(x_n))$ has a cluster point in $f(D)$. Since f is closed, (x_n) has a cluster point $w \in D$. Since $w \notin \operatorname{Int} M$, there exists $G \in \mathcal{G}$ such that $w \in G$. Since some $x_n \in G$, there exists j such that G is the jth term of $\alpha(x_n)$. There exists $m > \max(n, j)$ such that $x_m \in G$. Since x_m is not in the first $m - 1$ terms of $\alpha(x_n)$ this involves a contradiction. It is straightforward to show that if X is regular and M is a closed subset of X such that every collection of open sets covering $\operatorname{Fr}(M)$ includes a finite subcollection $\mathcal{F}$ such that $\bigcup \mathcal{F} \cup M$ is open, then $\operatorname{Fr}(M)$ is compact.

Lemma 3.8. *Suppose (X, σ) and (Y, τ) are topological spaces, Y is T_1, and f is a peripherally compact closed continuous mapping X onto Y. Suppose that $(\mathcal{C}_n)$ is a primitive sequence of X in X such that each $\mathcal{C}_n \subset \sigma$.*

Then there exists a sequence $(\mathcal{H}_n)$ of well-ordered subcollections of τ covering Y and sequences (M_n) and $(\mathcal{U}_n)$ of functions such that for each $n \in N$:

(P1) *The domain of M_n is $\mathcal{H}_n$ and its range is included in Y. For all $H \in \mathcal{H}_n$, $M_n(H)$ is a point of H not in any predecessor of H. If $y \in Y$ and $\{y\}$ is open then $\{y\} \in \mathcal{H}_n$ and $M_n(\{y\}) = y$.*

(P2) *If $y \in Y$, $j \in N$, $j < n$, and H and H' are the first elements of $\mathcal{H}_j$ and $\mathcal{H}_n$, respectively, that contain y then $H \supset H'$. If X is regular and T_0, $H \supset \overline{H}'$.*

(P3) *The domain of $\mathcal{U}_n$ is the collection of nondegenerate elements of $\mathcal{H}_n$ and its range is included in the collection of all finite subcollections of τ. If H is in the domain of $\mathcal{U}_n$, $\mathcal{U}_n(H)$ covers both $\operatorname{Fr}(f^{-1}(M_n(H)))$ and $f^{-1}(H)\backslash f^{-1}(M_n(H))$. Also each $U \in \mathcal{U}_n(H)$ contains $x \in \operatorname{Fr}(f^{-1}(M_n(H)))$ such that if $k < n$ and H' is the first element of $\mathcal{H}_k$ that contains $M_n(H)$ then $U' \in \mathcal{U}_k(H')$ and $x \in U'$ implies $U \subset f^{-1}(H') \cap U'$. If X is regular and T_0 then $\overline{U} \subset f^{-1}(H') \cap U'$. Furthermore, each $U \in \mathcal{U}_n(H)$ contains z such that the first element of $\mathcal{C}_n$ that contains z includes U.*

Proof. It will be assumed that both σ and the family Φ of all finite subcollections of σ are well ordered. Let $<_0$ denote a well ordering of Y such that if $y \in Y$ and $\{y\}$ is open then y precedes (in the $<_0$ sense) every element of Y which does not have the same property. Suppose there exist well-ordered collections $\mathcal{H}_1, \cdots, \mathcal{H}_k$, functions $M_1, \cdots, M_k$, and functions $\mathcal{U}_1, \cdots, \mathcal{U}_k$ satisfying (P1)–(P3) for all $n \leq k$ and let $<_{k-1}$ be a well ordering of Y. Define an ordering $<_k$ in Y as follows: If $y, y' \in Y$ and H and H' are the first elements of $\mathcal{H}_k$ that contain y and y' respectively, then $y <_k y'$ if and only if either (1) H precedes

H' in $\mathcal{H}_k$ or (2) $H = H'$ and $y <_{k-1} y'$. It may be seen that $\leq_k$ well orders Y. Let Y_k denote Y under this ordering. Now a sequence function g of type Y_k in τ is to be defined [5]. For $y \in Y_k$ let $s(y) = \{z \in Y_k : z <_k y\}$. Suppose $y \in Y_k$ and t: $s(y) \longrightarrow \tau$. If $\{y\}$ is open, let $g(t) = \{y\}$. If $\{y\}$ is not open, consider two cases. *Case* (i). *There exists* $z \in s(y)$ *such that* $y \in t(z)$. Let y' denote the first such z in Y_k and let $g(t) = t(y')$. *Case* (ii). *If* $z <_k y$ *then* $y \notin t(z)$. Let H_j denote the first element of $\mathcal{H}_j$ that contains y. Let M denote $\mathrm{Fr}(f^{-1}(y))$ and suppose $x \in M$. Let W denote the intersection of all sets of $\mathcal{U}_j(H_j)$ that contain x for $1 \leq j \leq k$ and let V denote the first element of $\mathcal{C}_{k+1}$ that contains x. There exists a first U_x in σ such that $x \in U_x \subset W \cap V \cap f^{-1}(H_k)$. If X is regular and T_0, it may also be required that $\overline{U}_x \subset W \cap V \cap f^{-1}(H_k)$. Since $\mathrm{Fr}(f^{-1}(y))$ is compact, there exists a first element $\mathcal{U}$ of Φ that is a subcollection of $\{U_x : x \in M\}$ covering M. Define $g(t)$ as $\{z \in Y : f^{-1}(z) \subset \bigcup \mathcal{U} \cup \mathrm{Int}\, f^{-1}(y)\}$. Since f is closed, $g(t) \in \tau$. By the transfinite recursion theorem there exists a unique function U: $Y_k \longrightarrow \tau$ such that $U(y) = g(U \mid s(y))$ for all $y \in Y_k$.

For every H in the range of U, let $M_{k+1}(H)$ denote the first element of Y_k such that $U(M_{k+1}(H)) = H$. Then M_{k+1} is a one-to-one function on the range of U into Y_k. This permits the definition of a well ordering $\preceq$ by $H \prec H'$ if and only if $M_{k+1}(H) <_k M_{k+1}(H')$ for H, H' in the range of U. Call the resulting well-ordered set $\mathcal{H}_{k+1}$.

1. If $y <_k M_{k+1}(H)$ then $M_{k+1}(H) \notin U(y)$. For if $M_{k+1}(H) \in U(y)$ there exists a first $z \in s(M_{k+1}(H))$ such that $M_{k+1}(H) \in U(z)$. Thus $H = U(M_{k+1}(H)) = g(U \mid s(M_{k+1}(H))) = U \mid s(M_{k+1}(H))(z) = U(z)$ contrary to the definition of $M_{k+1}(H)$.

2. If $y \in Y_k$ the first element H of $\mathcal{H}_{k+1}$ that contains y is $U(y)$. For $y \in U(y)$ since $g(U \mid s(y)) = U(y)$ contains y. If $H \prec U(y)$ and $y \in H$ then $y \in U(M_{k+1}(H))$ where $M_{k+1}(H) <_k y$. For $M_{k+1}(H) \leq_k y$ and $y \neq M_{k+1}(H)$ since $H \neq U(y)$. By definition of g, Case (i), $U(y) = U(z)$ for the first $z <_k y$ such that $y \in U(z)$. From the definition of z, $z = M_{k+1}(U(z)) \leq_k M_{k+1}(H)$. Thus $U(y) = U(z) \preceq H \prec U(y)$ which is a contradiction.

Suppose $y \in Y_k$ and $\{y\}$ is open. Then $U(y) = \{y\}$ and $M_{k+1}(\{y\}) = y$. From this and 2 it follows that (P1) is satisfied. To show that (P2) is satisfied suppose $y \in Y$ and H and H' are the first elements of $\mathcal{H}_k$ and $\mathcal{H}_{k+1}$ respectively that contain y. Let G be the first element of $\mathcal{H}_k$ that contains $M_{k+1}(H')$. Since $H' = U(M_{k+1}(H')) = g(U \mid s(M_{k+1}(H')))$ it follows from the definition of g that $G \supset H'$ (if X is regular and T_0, $G \supset \overline{H'}$). Therefore $y \in G$ so that $H \preceq G$. If $H \prec G$, then $y <_k M_{k+1}(H')$. Since 2 implies that $U(y) = H'$, this is impossible. Therefore $H = G$.

To show that (P3) holds, consider any nondegenerate element H of $\mathcal{H}_{k+1}$. There exists, by the definition of g, a first $\mathcal{U} \in \Phi$ such that $H = \{z \in Y : f^{-1}(z) \subset \bigcup \mathcal{U} \cup \mathrm{Int}\, f^{-1}(M_{k+1}(H))\}$. Let $\mathcal{U}_{k+1}(H)$ denote $\mathcal{U}$. Then $\mathcal{U}_{k+1}(H)$ is finite and covers $\mathrm{Fr}(f^{-1}(M_{k+1}(H)))$ and $f^{-1}(H) \setminus f^{-1}(M_{k+1}(H))$. If $U \in \mathcal{U}_{k+1}(H)$ then $U = U_x$

for some $x \in \mathrm{Fr}(f^{-1}(M_{k+1}(H)))$. If $j < k+1$ and H' is the first element of $\mathcal{H}_j$ that contains $M_{k+1}(H)$ and $U' \in \mathcal{U}_j(H')$ contains x then the definition of g, Case (ii), shows that $U_x = U \subset U' \cap V \cap f^{-1}(H')$ where V is the first element of $\mathcal{C}_{k+1}$ that contains x. In the regular T_0 case $\overline{U} \subset U' \cap V \cap f^{-1}(H')$. It follows that (P3) is satisfied. From the definition of g it should be clear how to obtain $\mathcal{H}_1$, M_1 and $\mathcal{U}_1$ such that the applicable parts of (P1)–(P3) hold. Thus, by mathematical induction, the sequences exist.

4. Proofs of theorems.

Proof of Theorem 2.1. Since the space X is assumed to be regular and T_0 the same is true of Y. Thus it suffices to show that Y has a monotonically contracting sequence satisfying the appropriate conditions of Lemmas 3.1 and 3.2 whenever X has one. Suppose X has a sequence $(\mathcal{G}_n)$ satisfying the appropriate conditions of Lemmas 3.1 or 3.2 accordingly as X satisfies λ_b, λ_c, β_b or β_c. There exists a primitive sequence $(\mathcal{C}_n)$ of X in X such that each $\mathcal{C}_n \subset \mathcal{G}_n$ by Lemma 3.3. By Lemma 3.8 there exists a sequence $(\mathcal{H}_n)$ with properties (P1)–(P3) relative to f and $(\mathcal{C}_n)$. Let $\mathfrak{W}_1$ denote $\mathcal{H}_1$. If $\mathfrak{W}_1, \cdots, \mathfrak{W}_n$ have been defined let $\mathfrak{W}_{n+1}$ denote the collection of all sets of the form $U \cap H$ where $H \in \mathcal{H}_{n+1}$, U is open, U contains a point of H not in any predecessor of H in $\mathcal{H}_{n+1}$, and for some $H' \in \mathcal{H}_n$, $\overline{U} \subset H'$. By an argument used in the proof of Lemma 3.6 it may be seen that each $\mathfrak{W}_n$ covers Y and if $y \in W \in \mathfrak{W}_n$ there exists $W' \in \mathfrak{W}_{n+1}$ such that $y \in W'$ and $\overline{W'} \subset W$. It is assumed that $X \neq \varnothing$ and that no element of $\mathcal{H}_1$ is empty. It follows that all elements of each $\mathfrak{W}_n$ are nonempty.

I. Suppose (W_n) is a representative of $(\mathfrak{W}_n)$ such that each $\overline{W}_{n+1} \subset W_n$. For each n there exists a first $H_n \in \mathcal{H}_n$ that includes a term of (W_n). By Lemma 3.4, (H_n) is decreasing. Suppose $n < k$. If H is the first element of $\mathcal{H}_n$ that contains $M_k(H_k)$ then $H \supset \overline{H}_k$. Therefore H does not precede H_n. Since $H_n \supset H_k$ it follows that $H = H_n$ and $H_n \supset \overline{H}_k$. If some $W_n = \{y\}$ for $y \in Y$ then $\bigcap\{W_n : n \in N\} = \{y\}$ and $\{W_n : n \in N\}$ is a base at $\{y\}$. Thus it may be assumed that no W_n is of this form.

II. Suppose β_c holds and there exists $y \in \bigcap\{W_n : n \in N\}$. Then $y \in J = \bigcap\{H_n : n \in N\}$. Let C_n denote $f^{-1}(M_n(H_n))$ and $\mathcal{U}_n$ denote $\mathcal{U}_n(H_n)$. Suppose $x \in \mathrm{Fr}(f^{-1}(y))$. With the use of (P3) and König's lemma [8] or [11, p. 47] it may be seen that there exists a sequence (U_n) such that, for all n, $\overline{U}_{n+1} \subset U_n \cap f^{-1}(H_n)$ and $x \in U_n \in \mathcal{U}_n$. By (P3) for each n there is a first $A_n \in \mathcal{C}_n$ that includes a term of (U_n) and by Lemma 3.4, (A_n) is decreasing. It follows from Lemma 3.5 and the fact that each $\mathcal{C}_n \subset \mathcal{G}_n$ that (U_n) has property B. For each n there exists $x_n \in U_n \cap C_n$. The sequence (x_n) has a cluster point $u \in \bigcap\{U_n : n \in N\}$. Therefore $f(u) \in \bigcap\{f(U_n) : n \in N\} \subset J$. Let C denote $f^{-1}(f(u))$. Suppose W is open and $W \supset J$. Then $C \subset f^{-1}(W)$. For each $z \in \mathrm{Fr}(C)$, with the use of König's lemma, (P3), and

a representative of $(\mathcal{C}_n)$ as above, there exists a sequence (V_n) with property B such that each $V_n \in \mathfrak{U}_n$, $\overline{V}_{n+1} \subset V_n \subset f^{-1}(H_n)$, and $z \in \bigcap\{V_n: n \in N\} \subset f^{-1}(W)$. Then for each $z \in \mathrm{Fr}(C)$ there exists j and $U \in \mathfrak{U}_j$ such that $z \in U \subset f^{-1}(W)$. Since $\mathrm{Fr}(C)$ is compact there exist k and a subcollection $\mathfrak{D}$ of $\mathfrak{U}_1 \cup \cdots \cup \mathfrak{U}_k$ such that $\mathrm{Fr}(C) \subset \bigcup \mathfrak{D} \subset f^{-1}(W)$. Let H denote $\{y \in Y: f^{-1}(y) \subset \bigcup \mathfrak{D} \cup \mathrm{Int}(C)\}$. Then H is open and $u \in f^{-1}(H)$. Therefore, for some $j > k$, $x_j \in f^{-1}(H)$ and thus $C_j \subset \bigcup \mathfrak{D} \cup \mathrm{Int}(C)$. If $U \in \mathfrak{U}_j$, U contains $x \in \mathrm{Fr}(C_j)$ such that if $n < j$ any element of $\mathfrak{U}_n$ which contains x includes U. Thus $\bigcup \mathfrak{U}_j \subset \bigcup \mathfrak{D}$ and $f^{-1}(H_j) \subset \bigcup \mathfrak{U}_j \cup C_j \subset f^{-1}(W)$. It follows that any open set which includes J includes some H_k. Suppose M is an infinite subset of J. If M includes an infinite subset of $\{M_n(H_n): n \in N\}$ then M has a limit point as follows from the fact that (x_n) has a cluster point. If M contains no such subset then, for each n, $f^{-1}(M)$ includes an infinite subset of $f^{-1}(H_n)\backslash\{C_k: k \in N\}$. Therefore, for each n, the set $\mathfrak{U}'_n = \{U \in \mathfrak{U}_n: |f(U) \cap M| \geq \aleph_0\}$ is not empty. By König's lemma there exists a sequence (U_n) such that each $\overline{U}_{n+1} \subset U_n \in \mathfrak{U}'_n$. It follows as above that $\bigcap\{U_n: n \in N\}$ has property B. There exists a sequence (y_n) of distinct points such that each $y_n \in M \cap f(U_n)$. If $y_n = f(z_n)$, where $z_n \in U_n$, there is a limit point z of $\{z_n: n \in N\}$ belonging to $\bigcap\{U_n: n \in N\}$. Therefore $f(z)$ is a limit point of M, so that J is countably compact and thus has property B. It follows from Lemma 3.5 that $\bigcap\{W_n: n \in N\}$ is countably compact and has property B. Therefore Y satisfies β_c if X does. Suppose X satisfies β_b and $\mathfrak{A}$ is a collection of open sets covering J. If $x \in f^{-1}(J)\backslash\{C_k: k \in N\}$ there exists a sequence (U_n) such that $x \in U_{n+1} \subset \overline{U}_{n+1} \subset U_n$ and $U_n \in \mathfrak{U}_n$ for all n. Each $\bigcap\{U_n: n \in N\}$ is a compact subset of $f^{-1}(J)$. Thus $f^{-1}(J)\backslash\{C_k: k \in N\}$ is covered by countably many compact subsets of $f^{-1}(J)$. It follows that $f^{-1}(J)$ is covered by countably many elements of $f^{-1}(\mathfrak{A})$. Therefore J has the Lindelöf property and, since it is T_1 and countably compact, it is compact. Thus $\bigcap\{W_n: n \in N\}$ is compact so Y satisfies β_b.

III. If $\lambda_c(\lambda_b)$ holds, then, using part of the argument of II, there exists $y \in \bigcap\{W_n: n \in N\}$. Then the argument of II shows that $\bigcap\{W_n: n \in N\}$ has property B (and is compact in case λ_b), and therefore Y satisfies $\lambda_c(\lambda_b)$.

Proof of Theorem 2.3. If X is a completely regular T_0 μ-space it satisfies β_b by Lemma 3.6. Therefore Y satisfies β_b by Theorem 2.1. Since Y is completely regular and T_0 it is a μ-space by Lemma 3.6. A similar argument proves the second part of the theorem.

Proof of Theorems 2.2 and 2.4. Under the hypothesis on f, X, and Y it follows from Lemma 3.7 that f is peripherally compact. Thus the results follow from Theorems 2.1 and 2.3.

Proof of Theorems 1.1 and 1.2. These theorems are corollaries to the propositions of I and II of the introduction and Theorem 2.1.

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